ISRAEL JOURNAL OF MATHEMATICS 87 (1994), 257–273

POLYNOMIAL REFLEXIVITY IN BANACH SPACES*

BY

JEFF D. FARMER

Department of Mathematics
University of Missoure, Columbia, MO 65211, USA

ABSTRACT

We ask when the space of N-homogeneous analytic polynomials on a Banach space is reflexive. This turns out to be related to whether polynomials are weakly sequentially continuous, and to the geometry of spreading models. For example, if these spaces are reflexive for all N, no quotient of the dual space may have a spreading model with an upper q-estimate, and every bounded holomorphic function on the unit ball has a Taylor series made up of weakly sequentially continuous polynomials (we assume the approximation property). Alencar, Aron and Dineen [AAD] gave the first example of some properties of a polynomially reflexive space (using T^*, the original Tsirelson space); we show that these properties and others are shared by all polynomially reflexive spaces.

Introduction

In this paper we examine the concept of polynomial reflexivity of a Banach space. In doing so, we bring together a number of disparate results concerning polynomials on Banach spaces, make connections with the theory of bounded analytic functions on the unit ball of a Banach space and relate some geometric properties of spreading models to hereditary polynomial reflexivity.

The paper is organized as follows. In Section 1 we consider the relationship between a hereditary form of polynomial reflexivity and geometric properties of spreading models of either the space or its dual. In Section 2 we define a form

* This paper forms a portion of the Ph. D. dissertation of the author, under the supervision of W. B. Johnson.
Received October 21, 1992 and in revised form May 25, 1993

of polynomial reflexivity indexed by the positive integers and relate it to some properties introduced in [CCG].

It has been observed by Alencar, Aron and Dineen [AAD] that in the case of Tsirelson's space, T^* (see [CS] for the definition), the spaces of polynomials $\mathcal{P}_N(X)$ are reflexive for all values of N. This is then used to show that the space of entire functions on T^* with the Nachbin topology is reflexive. Alencar, Aron and Fricke [AAF] have shown that any operator from N-fold projective tensor products of T^* into ℓ_p is compact. In Section 3 we show how these results can be generalized using the concept of polynomial reflexivity and that of a polynomially Schur space.

In Section 4 we consider the structure of the algebra of bounded analytic functions on the unit ball of a Banach space in the context of polynomial reflexivity, showing, for example, that tightness of the algebra implies polynomial reflexivity. We also give a partial answer to a question of Davie and Gamelin [DG] concerning the identification of evaluations in the spectrum of $\mathcal{H}^\infty(B)$.

1. Polynomial reflexivity and geometric properties of spreading models

We begin with some terminology and notation. Let X be a complex infinite-dimensional Banach space. An **N-homogeneous analytic polynomial** on X is the restriction to the diagonal of an N-linear form on the N-fold Cartesian product of X with itself, or equivalently, a linear functional on the N-fold projective tensor product of X with itself. Indeed, given an N-homogeneous analytic function P on X, one obtains an N-linear form on X by taking the Nth derivative and dividing by $N!$. The form is related to the polynomial by the polarization formula:

$$A_P(x_1,\ldots,x_n) = \mathrm{Avg}\{\epsilon_i = \pm 1\}(\Pi\epsilon_i)P\Big(\sum_{i=1}^{n}\epsilon_i x_i\Big).$$

The form A_P is clearly symmetric (invariant under permutations of the coordinates). Likewise any bounded symmetric N-linear form will give rise to an N-homogeneous analytic polynomial. Such a form can be linearized by taking the projective tensor product of X with itself N times and extending the form to a linear functional on this tensor product. The subspace of symmetric linear functionals is the dual of the symmetric N-fold projective tensor product, which is a complemented subspace of the N-fold projective tensor product. The projection is given by extending the following map linearly:

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \to \frac{1}{n!} \sum_{\pi \in S_n} (x_{\pi_1} \otimes x_{\pi_2} \otimes \cdots \otimes x_{\pi_n}).$$

We denote the symmetric projective tensor product by $\widehat{\otimes}_s^N X$. The N-linear form A_P associated with P can now be considered a linear functional on $\widehat{\otimes}_s^N X$. The supremum norm of the polynomial is related to that of the linear functional as follows:

$$||P|| \leq ||A_P|| \leq \frac{N^N}{N!} ||P||.$$

If we call the space of polynomials $\mathcal{P}_N$ the above simply says that $\mathcal{P}_N$ is isomorphic to $(\widehat{\otimes}_s^N X)^*$. Since for our purposes the index N will be fixed, we will suppress reference to this isomorphism and use the same label for a polynomial and its associated symmetric linear functional. More details about the above relationships may be obtained from [M] or [R1].

If the space $\mathcal{P}_N$ of N-homogeneous analytic polynomials on a complex Banach space X is reflexive, we will simply say that X is **$\mathcal{P}_N$-reflexive**. If this is true for every N then we will say that X is **polynomially reflexive**. Polynomial reflexivity passes to quotients, because polynomials always lift from quotients, though they do not in general extend from subspaces. In the presence of the approximation property, polynomial reflexivity is equivalent to weak sequential continuity of polynomials [see AAD], and it is this property which we can more readily relate to the geometry of spreading models on the space.

Let us recall the concept of a spreading model, the construction of which is due to Brunel and Sucheston [BS1].

Finite versions of Ramsey's Theorem allow that given any property of n-tuples of elements from a sequence, one can pass to a subsequence with the property that all n-tuples formed from the subsequence share the property or else all fail it. By using the size of the norm of a sum of n elements as the property one can, by repeatedly applying the theorem, approximately stabilize the norm (to within any desired ϵ_n) of any finite combination as long as many of the beginning terms are thrown away. More precisely we have the following fact (see [B] or [BS1]).

PROPOSITION 1.1: *Let (f_n) be a bounded sequence with no norm-Cauchy subsequence in a Banach space X. Then there exists a subsequence (e_n) of (x_n) and a norm L on the vector space S of finite sequences of scalars such that for* *all* $\epsilon > 0$ *and all* $a \in S$ $(\max|a_i| \leq 1)$ *there exists* $k \in N$ *such that for all* $k < k_1 < k_2 < \cdots < k_M$ *we have*

$$\left| \left\| \sum a_i e_{k_i} \right\| - L(a) \right| < \epsilon.$$

The completion of $[e_i]$ (call it F) under the norm L is called a **spreading model** for the sequence (e_n). The reason for the terminology is that the sequence (e_n) is invariant under spreading with respect to the norm F, that is to say, for every finite sequence of scalars (a_i) and every subsequence σ of the natural numbers

$$\left\| \sum_{i=1}^{M} a_i e_i \right\|_F = \left\| \sum_{i=1}^{M} a_i e_{\sigma(i)} \right\|_F .$$

Thus any norm estimate satisfied by sequences in the spreading model will be approximately satisfied for sequences of finite length to any desired degree provided we go out far enough in the sequence (x_n). If the original sequence was weakly null then the resulting sequence will be unconditional; that is to say, we have the following (Lemma 2 of [B], or see [BS2]).

PROPOSITION 1.2: *If* (x_n) *is weakly null, then the sequence* (e_n) *is unconditional in* F *with unconditional constant at most 2.*

Now we state the main theorem of the section.

THEOREM 1.3: *Suppose* X *is a reflexive Banach space. Then,*

(i) *If no spreading model built on a weakly null sequence has a lower* q*-estimate for any* $q < \infty$ *then any polynomial on any subspace of* X *will be weakly sequentially continuous at the origin. Thus every subspace of* X *with the approximation property will be polynomially reflexive.*

(ii) *Suppose every polynomial on every subspace of* X *is weakly sequentially continuous at the origin. Then no quotient of* X^* *has a weakly null sequence with an upper* p*-estimate for any* $p > 1$.

In [AAD] it was shown that Tsirelson's space is polynomially reflexive using essentially the fact that T^* has a basis whose only spreading model is c_0. What is actually needed is a spreading model having no lower q-estimate. Recall that an unconditional basis (x_n) has a lower q-estimate (for some $q < \infty$) if there exists a constant $c > 0$ such that for any scalars (a_n),

$$\left\| \sum_{n=1}^{k} a_n x_n \right\| \geq c \left(\sum_{n=1}^{k} |a_n|^q \right)^{1/q} .$$

We will also use the equivalence shown in [AAD, proof of Proposition 4] that for a reflexive space which has the approximation property, $\mathcal{P}_M$ is reflexive if and only if every element of $\mathcal{P}_M$ is weakly sequentially continuous at the origin. We now proceed to the proof of the first part of the theorem; the proof of the second part is deferred until Section 4 (it follows from Theorem 4.1), where we relate polynomial reflexivity to some properties of algebras of analytic functions and the topologies they generate on the space.

Proof of (i): Suppose that $P \in \mathcal{P}_M$ is not weakly sequentially continuous at the origin. Then there is a (without loss of generality normalized) weakly null sequence on which P is bounded away from zero in modulus, and we can by 1.1 build a spreading model on a subsequence of it. Since the underlying sequence is weakly null, the spreading model will be unconditional, by 1.2. Passing to the subsequence and relabeling we have that for some $\epsilon > 0$,

$$|P(x_i)| > \epsilon \quad \forall i \geq 1.$$

Now notice that for fixed x and y, $P(x + \lambda y) = \sum_0^M a_j \lambda^j$ is a polynomial of degree M on the complex numbers with constant coefficient $a_0 = P(x)$ and M-th coefficient $a_M = P(y)$. This can be seen for example by writing out the M-linear form associated with P:

$$A_P(x + \lambda y, x + \lambda y, \ldots) = A_P(x, x, \ldots) + \lambda A_P(y, x, \ldots) + \cdots + \lambda^M A_P(y, y, \ldots).$$

We then have

$$|P(x)|^2 + |P(y)|^2 = |a_0|^2 + |a_M|^2 \leq \left(\sum_0^M |a_j|^2\right) \leq \sup_{|\lambda| \leq 1} \left|\sum_0^M a_j \lambda^j\right|^2 = P(x + \eta y)^2$$

for an appropriate choice of η with $|\eta| = 1$.

If we apply the above fact repeatedly to any subset A of the (x_i) of size 2^l, we can find $|\lambda_i| = 1, \; i = 1, \ldots, 2^l$ so that

$$\|P\| \left\| \left(\sum_{i \in A} \lambda_i x_i\right) \right\|^M \geq \left| P\left(\sum_{i \in A} \lambda_i x_i\right) \right| \geq \epsilon (\sqrt{2})^l.$$

We can use this to obtain norm estimates for the spreading model if we confine ourselves to considering sets which begin far enough out in the sequence; since the above is true for all subsets of size 2^l, we may say

$$\left\| \sum_{i \in A} \lambda_i x_i \right\| \leq 2 \left\| \sum_{i \in A} \lambda_i x_i \right\|_F$$

where F is the spreading model. Using the unconditionality of the spreading model (we know it has unconditional constant ≤ 2), we have

$$\Big\|\sum_{i\in A}\lambda_i x_i\Big\|_F \leq 2\Big\|\sum_{i\in A}\epsilon_i\lambda_i x_i\Big\|_F$$

for all choices $|\epsilon_i| = 1$.

Combining the above we obtain

$$\Big\|\sum_{i\in A}\epsilon_i\lambda_i x_i\Big\|_F \geq C(2^l)^{1/2M}$$

where $C = \frac{1}{4}(\frac{\epsilon}{\|P\|})^{1/M}$.

Although we have shown this for sets of size 2^l, we can now modify the constant and obtain a similar statement for sets of any size. Now choose $q > 2M$ and by applying the following standard lemma (with $r = 2M$) we will have an unconditional spreading model with a lower q-estimate.

LEMMA 1.4: *Suppose (x_i) is a normalized unconditional basic sequence with the property that for subsets A of the integers,*

$$C\|\sum_{i\in A} x_i\| \geq |A|^{1/r}.$$

Then for every $q > r$ there is another constant C' so that for all sets B

$$C'\|\sum_{i\in B} a_i x_i\| \geq \big(\sum_{i\in B}|a_i|^q\big)^{1/q}.$$

Proof: Let

$$D = \big(\sum_{i=1}^{\infty} 2^{(-i-1)(q-r)}\big)^{1/r} \quad\text{and}\quad A_n = \{i \in B | 2^{-n} \geq a_i > 2^{-n-1}\}$$

where B is a subset of $\mathbf{N}$ and K is the unconditional constant. Assume that

$\max\{|a_i|\} = 1$ and write

$$\begin{aligned}
2KDC\|\sum_{i\in B} a_i x_i\| =& 2KDC\|\sum_{n=0}^{\infty}\sum_{i\in A_n} a_i x_i\| \\
&\geq DC \sup_n 2^{-n-1}\|\sum_{i\in A_n} x_i\| \\
&\geq \sup_n\{2^{-n-1}|A_n|^{1/r}\}\Big(\sum_{i=0}^{\infty} 2^{(-i-1)(q-r)}\Big)^{1/r} \\
&\geq \left(\sum_{n=0}^{\infty}(2^{-n-1})^r|A_n|\ 2^{(-n-1)(q-r)}\right)^{1/r} \\
=& \left(\sum_{n=0}^{\infty}(2^{-n-1})^q|A_n|\right)^{1/r} \\
&\geq 2^{1-q/r}\left(\sum_{n=0}^{\infty}(2^{-n-1})^q|A_n|\right)^{1/q} \\
&\geq 2^{-q/r}\Big(\sum_{i\in B}|a_i|^q\Big)^{1/q}
\end{aligned}$$

which completes the proof of the lemma by giving $C' = 2^{\frac{q}{r}+1}KDC$. Since the condition of having no unconditional spreading model with a lower q-estimate is hereditary we see that every polynomial on every subspace of X will be weakly sequentially continuous at the origin. ∎

Remark: It is worth noting that we have actually proved the following: If X is reflexive with the approximation property and has no spreading model with a lower q-estimate (for some fixed q), then X is $\mathcal{P}_M$-reflexive for all $M < q/2$.

2. $\mathcal{P}_N$-Reflexivity

In this section we note the following equivalences.

PROPOSITION 2.1: *The following are equivalent for any Banach space X, and any fixed (extended) positive integer m.*

(i) *For all $N \leq m$, the restriction of an N-homogeneous polynomial to a weakly compact subset of X is weakly continuous.*

(ii) *For all $N \leq m$, any N-homogeneous polynomial on X is weakly sequentially continuous.*

(iii) *For all* $N \leq m$, *the function* θ *taking each* $x \in X$ *to the corresponding diagonal element in* $X\widehat{\otimes}\cdots\widehat{\otimes}X$ (N times) *is weak-to-weak sequentially continuous.*

(iv) *For all* $N \leq m$, θ *is weak-to-weak sequentially continuous at the origin, that is, if* $\{x_k\}_{k=1}^{\infty}$ *is a weakly null sequence in* X , *then* $\{x_k \otimes x_k \otimes \cdots \otimes x_k\}$ *is a weakly null sequence in* $X\widehat{\otimes}X\widehat{\otimes}\cdots\otimes X$ (N times).

(v) *For all* $N \leq m$, *the restriction of* θ *to any weakly compact subset of* X *is weak-to-weak continuous.*

These equivalences were proved in [CCG] (for the case $m = \infty$) and so we will only sketch the proof here.

It is clear that (i) is merely a restatement of (v) and (ii) of (iii). That (i) implies (ii) is obvious, and that (ii) implies (i) follows from 7.2 of [CCG] which essentially points out that because of the Eberlein–Smulian Theorem it is enough to check weak continuity using sequences if one is restricted to a weakly compact set. That (iii) implies (iv) is trivial, while the converse follows by writing $P(x_n)$ as $P(x-(x-x_n))$ and expanding the m-linear form associated with P as we did in the proof of Theorem 1.3. One can then assume by induction on N that the mixed terms will go to zero and this finishes it. ∎

Now Ryan [R1] characterized the reflexivity of $\mathcal{P}_N$ by the weak continuity of each element of $\mathcal{P}_N$ on the unit ball, and [AAD] showed that it is enough to check this at the origin (all this in the context of reflexive spaces with the approximation property). Thus it is clear that in this context the above equivalences actually characterize $\mathcal{P}_N$-reflexivity.

3. Tensor products and operators on polynomially reflexive spaces

We next generalize some of the results of Alencar, Aron and Fricke [AAF], in which it is shown that projective tensor products of Tsirelson space are reflexive [AAF, Proposition 1] and operators from these products into ℓ_p spaces ($1 \leq p < \infty$) are all compact [AAF, Corollary 8]. These results of [AAF] can be subsumed and generalized by the following.

We first recall from [FJ] that a Banach space is **polynomially Schur** (**$\mathcal{P}_N$-Schur**) if whenever a sequence converges to zero against every polynomial (respectively, every N-homogeneous polynomial) then it must tend to zero in norm.

THEOREM 3.1: *Suppose X is polynomially reflexive ($\mathcal{P}_N$-reflexive) and has the approximation property, and suppose that Y is polynomially Schur (respectively, $\mathcal{P}_N$-Schur). Then every bounded linear operator from X to Y is compact, and if in addition Y is reflexive, then $B(X,Y) = K(X,Y)$ is also reflexive.*

Proof: Suppose without loss of generality that (x_n) is a bounded sequence in the ball of X which converges weakly to zero. Then (x_n) converges polynomially (N-polynomially) to zero, and therefore so does $T(x_n)$ (where T is any operator). But since Y is polynomially Schur ($\mathcal{P}_N$-Schur), $T(x_n)$ must converge to zero in norm. The second statement follows by standard duality arguments (see [DU, p. 247]). ∎

Note that (as a consequence of 2.1 (iv)) the symmetric projective N-fold tensor product of a polynomially reflexive space is also polynomially reflexive. In fact, the N-fold projective tensor product of a $\mathcal{P}_{NM}$-reflexive space is $\mathcal{P}_M$-reflexive. This fact allows us to extend 3.1 to the context of vector-valued polynomials.

THEOREM 3.2: *Suppose X is polynomially reflexive ($\mathcal{P}_{MN}$-reflexive) and has the approximation property, and suppose that Y is polynomially Schur (respectively, $\mathcal{P}_N$-Schur). Then every bounded (vector-valued) M-homogeneous polynomial from X to Y is compact.*

Proof: The proof follows the same lines as 3.1, using the above remark. ∎

Since T^* is polynomially reflexive and all ℓ_p spaces are polynomially Schur ($1 \leq p < \infty$) [CCG] we now obtain [AAF, Proposition 8] as a corollary.

The final result of [AAF] will also generalize.

PROPOSITION 3.3: *Let X be a Banach space with an unconditional basis. Then X is polynomially reflexive if and only if for every polynomially Schur space Y, $(H(X,Y),\tau)$ (the space of all holomorphic functions from X to Y with the Nachbin topology) is reflexive.*

Proof: We simply check that in [AAF, Corollary 9] the only fact used about $(H(T^*,\ell_p),\tau)$ is that the spaces $\mathcal{P}_N(T^*)$ are all reflexive. ∎

4. Bounded analytic functions on the unit ball

In this section we relate polynomial reflexivity to some questions about algebras of analytic functions defined on the unit ball B of a dual Banach space X. For our purposes, $\mathcal{A}(B)$ will be the uniform closure of the algebra generated by the weak-star continuous linear functionals, and $\mathcal{H}^\infty(B)$ will be the algebra of bounded analytic functions on the open unit ball.

Carne, Cole, and Gamelin have considered various topologies induced on the ball B by considering it as a subset of $\mathcal{A}(B)^*$ via evaluations. The topologies induced by the norm, weak and weak-star topologies of $\mathcal{A}(B)^*$ can then be compared to the usual topologies on the ball. In fact, $\mathcal{P}$-Schur spaces are of interest for exactly this reason, since (at least if X has the metric approximation property) every polynomial on B is the restriction of some element of $\mathcal{A}(B)^{**}$ to the ball (i.e. to the point masses); thus for these $\mathcal{P}$-Schur spaces $\sigma(\mathcal{A}(B)^*, \mathcal{A}(B)^{**})$ convergence of a sequence of point masses will force the points to converge in norm in B.

Our notation in general will follow [CCG]. We will denote by $\mathcal{A}_{pb}(B)$ the algebra of functions pointwise approximable by bounded nets of functions from $\mathcal{A}(B)$ and recall from [CCG] that $\mathcal{A}_{pb}(B) \subset \mathcal{H}^\infty(B)$ is always true; the reverse inclusion holds if X has the metric approximation property. The set $\mathcal{A}_{pb}(B)$ is important because it is exactly the set of all elements of $\mathcal{A}(B)^{**}$ restricted to B.

One question raised in [CCG] concerned whether the algebra $\mathcal{A}(B)$ is "tight." There it was conjectured that in the case of an infinite-dimensional Banach space X, $\mathcal{A}(B)$ is never tight. Indeed, those authors showed that for many classes of Banach spaces this is so. The ball algebra on a polynomially Schur space cannot be tight; neither can that of a nonreflexive space, nor a space which has ℓ_p as a quotient; Jaramillo and Prieto [JP] also conclude that for a separable space tightness will require that every point of the unit sphere be complex extreme.

Tightness is an analytic function theoretic concept that we will not define at the moment; what concerns us here is that a consequence of tightness is the agreement of the $\sigma(\mathcal{A}(B)^*, \mathcal{A}(B)^{**})$ topology with the weak-star topology on balls of radius less than one. In fact, it is this agreement of topologies from which many of the results in [CCG] concerning tightness are deduced; we will focus on this agreement itself.

A consequence of our first theorem will be that the only class of Banach spaces for which the ball algebra may be tight are the polynomially reflexive ones (or at least those with all polynomials weakly sequentially continuous); this will be found to subsume and extend several of the aforementioned results. (Part of the following theorem was proved by participants in an informal seminar organized by W. B. Johnson.)

THEOREM 4.1: *Consider the following statements, for a reflexive Banach space* X*:*

(i) X^* *has an unconditional spreading model (built on a weakly null sequence) with an upper p-estimate for some* $p > 1$.

(ii) *The* $\sigma(\mathcal{A}(B)^*, \mathcal{A}(B)^{**})$ *topology does not agree with the weak topology on* rB, *for* $r < 1$.

(iii) *There is an* N*-homogeneous polynomial (for* $N > p'$*) on* X *which is not weakly sequentially continuous.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). *If, in addition,* X *has the approximation property, then* (iii)$\Rightarrow$(ii) *and (as stated previously)* (iii) *is equivalent to the statement that* X *is not* $\mathcal{P}_N$*-reflexive.*

Proof that (i) implies (ii): Let $(e_i)^*$ be the sequence on which the spreading model is built. We can assume by passing to a subsequence that it is basic. Call its closed span E^* with predual E, a quotient of X, with π the natural quotient map, and (e_i) the dual basis in E. Find (x_i) in X with $\pi(x_i) = e_i$ with (x_i) weakly null (by passing to a subsequence if necessary). Choose $N > p'$ and define

$$P_k = \sum_{i=1}^{k} {e_i}^{*N}.$$

It is a consequence of the proof of 3.3 of [FJ] that the norms of these N-homogeneous polynomials are uniformly bounded (although we may need to pass once more to a subsequence, it does not affect the result). Consider the sequence $\{P_k \circ \pi\}_{k=1}^{\infty} \in \mathcal{A}(B)$ as a subset of $\mathcal{A}(B)^{**}$ and take Q to be any weak-star limit point. Let $\mathcal{U}$ be any ultrafilter on the integers and calculate

$$Q\Big(\frac{r}{2}x_i\Big) = \lim_{k\in U \subset \mathcal{U}} P_k \circ \pi\Big(\frac{r}{2}x_i\Big) = \lim_{k\in U\subset \mathcal{U}} P_k\Big(\frac{r}{2}e_i\Big) = \Big(\frac{r}{2}\Big)^N.$$

In fact, this element Q, considered as a function on the ball, is a bounded polynomial, giving (iii) as well as (ii).

Proof that (ii) implies (iii): Since X is reflexive we need only consider weak sequential convergence. Suppose that (x_i) converges weakly to $x \in rB$. We will show that if all polynomials are weakly sequentially continuous, then the sequence converges $\sigma(\mathcal{A}(B)^*, \mathcal{A}(B)^{**})$. Consider $F \in \mathcal{A}(B)^{**}$. Again by considering F as a function on the point masses we can think of $F\big|_B$ as an element of $\mathcal{H}^\infty(B)$ (we will write it also as F). We can write its Taylor series which converges pointwise on B:

$$F = \sum_{i=1}^{\infty} f_i \quad \text{with} \quad \|f_i\| \leq \|F\|.$$

For $r < 1$ and $\epsilon > 0$ we can calculate

$$\|f_i\big|_{rB}\| = \sup_{x \in rB} |f_i(x)| = \sup_{x \in B} |f_i(rx)| = r^i \|f_i\|.$$

Hence,

$$\begin{aligned}\|\sum_{i=k}^{\infty} f_i\Big|_{rB}\| &\leq \sum_{i=k}^{\infty} \|f_i\Big|_{rB}\| = r^k \sum_{i=k}^{\infty} r^{i-k}\|f_i\| \\ &\leq r^k \sum_{j=0}^{\infty} r^j \|F\| \leq r^k \left(\frac{\|F\|}{1-r}\right) < \epsilon\end{aligned}$$

by choosing k large enough. Since all polynomials (in particular the f_i) are weakly sequentially continuous, we can have

$$\big|F(x_j) - F(x)\big| \leq \Big|\sum_{i=1}^{k-1} f_i(x_j) - \sum_{i=1}^{k-1} f_i(x)\Big| + 2\epsilon < 3\epsilon$$

by picking j large, and so we are done.

To prove the last statement we use the fact [DU] that if a reflexive space has the approximation property, then both it and its dual have the metric approximation property. So suppose that we have a polynomial that is not weakly sequentially continuous at the origin, that is, let (x_i) converge weakly to zero with $P(x_i)$ bounded away from zero. Let ϵ_i go to zero, and let T_i be finite rank operators which approximate the identity on $(x_k)_{k=1}^{i}$ within a factor of ϵ_i, with norms of the T_i uniformly bounded. Then $P \circ T_i$ are finite-type polynomials and so are clearly in $\mathcal{A}(B)$. Let F be any weak-star limit of the set $\{P \circ T_i\}_{i=1}^{\infty}$ in $\mathcal{A}(B)^{**}$. Since P is uniformly continuous on bounded sets (say with modulus δ), we can see that given k and δ' we can find $i > k$ so that

$$|F(x_k) - PT_i(x_k)| \leq \delta(\epsilon_i) + \delta'$$

which implies that $F(x_k) \not\to 0$. ∎

We can now state the following corollary of this result.

COROLLARY 4.2: *Suppose X has the approximation property and $\mathcal{A}(B)$ is tight. Then $\mathcal{P}_N(X)$ is reflexive for all $N \geq 1$.*

Although T^* is an example of a polynomially reflexive space, Jaramillo and Prieto [JP] show that the ball algebra of T^* is not tight. This results from having points on the sphere which are not complex extreme, however, and if we consider (for example) the 2-convexification of T^*, we have a space which is still polynomially reflexive but for which the question of tightness is still open. Of course it may be that the ball algebra is never tight if X is infinite-dimensional.

We now turn to characterizations of polynomially reflexive spaces in terms of the algebra $\mathcal{H}^\infty(B)$. First we recall that the spectrum of a uniform algebra is the topological space of complex homomorphisms on the space; the topology is the relative weak-star topology from the unit sphere of the dual space of the algebra. The spectrum $\mathcal{M}_{\mathcal{H}^\infty(B)}$ fibers over the unit ball of X^{**} by restricting each complex homomorphism to the elements of X^*.

PROPOSITION 4.3: *The following are equivalent for any dual Banach space X whose dual has the approximation property.*

(i) *X is polynomially reflexive.*
(ii) *$\widehat{\otimes}_s^N X$ is reflexive for all N.*
(iii) *Every f in $\mathcal{H}^\infty(B)$ has a homogeneous expansion (i.e., Taylor series) whose partial sums are all weak-star continuous.*
(iv) *Every $f \in \mathcal{H}^\infty(B)$ is weak-star continuous on rB for any $0 < r < 1$.*

Remark: Of course, in the case where these equivalent conditions hold, the weak star topology coincides with the weak topology, so that the above statements (iii) and (iv) still hold for the weak topology.

Proof: (i) and (ii) are trivially equivalent; (iii) and (iv) are equivalent because the Taylor series converges uniformly on balls of radius less than 1; the fact that (i) and (iii) are equivalent follows from the remark at the end of Section 2. ∎

To consider some other characterizations of polynomially reflexive spaces, we need to recall some results from [DG] about the polynomial-star topology. We say that a net $\{x_\alpha\}$ in B^{**} (the unit ball of X^{**}) converges to x^{**} in the polynomial-star topology if $P(x_\alpha) \to \widehat{P}x^{**}$ for every polynomial P on X, where $\widehat{P}$ is the canonical extension of P to the ball of the second dual. This extension was first defined by Aron and Berner [AB], and was shown to be defined on the entire ball of the second dual by Davie and Gamelin. It is constructed by writing the linear form associated with the polynomial and extending it by weak-star continuity one coordinate at a time. Davie and Gamelin [DG] then showed that the unit ball of any Banach space X is polynomially-star dense in the ball of X^{**}. In fact they proved that if x^{**} is a weak-star cluster point of any set S in X, then it is in the polynomial-star closure of the convex hull of S. They also showed that any bounded analytic function on the unit ball of X has a canonical extension to a bounded analytic function on the unit ball of X^{**} with the same norm; this extension is formed by extending the terms of the Taylor series (which are homogeneous polynomials) as described above. Thus every point x^{**} in the ball of X^{**} gives a homomorphism in the spectrum via evaluation of the canonical extension. A question of Davie and Gamelin was how one can distinguish this evaluation from the other homomorphisms in the fiber over x^{**}. We can give a partial answer to this question.

PROPOSITION 4.4: *Suppose a net $\delta_{x_\alpha} \to \phi \in \mathcal{M}$ in the Gelfand topology and $x_\alpha \to z^{**}$ in the weak-star topology of X^{**}, with $||x_\alpha|| < 1 - \epsilon$ for some $\epsilon > 0$. Then $x_\alpha \to z^{**}$ in the polynomial-star topology if and only if $\phi = \delta_{z^{**}}$.*

Proof: We first prove that the condition is sufficient. We may assume without losing anything that the x_α are in X. For every $f \in \mathcal{H}^\infty(B)$ there exists a sequence of polynomials P_i so that $\widehat{P}_i$ will converge uniformly to $\widehat{f}$ on $(||z^{**}|| + \frac{\epsilon}{2})B^{**})$. We then have

$$\phi(f) = \lim_\alpha f(x_\alpha) = \lim_\alpha \lim_i P_i(x_\alpha)$$

which by uniform convergence of the P_i equals

$$\lim_i \lim_\alpha P_i(x_\alpha) = \lim_i \widehat{P}_i(z^{**}) = \hat{f}(z^{**}).$$

On the other hand, suppose that the convergence is not polynomial-star and choose a polynomial P for which the convergence fails. Write

$$\delta_{z^{**}}(P) = \widehat{P}(z^{**}) \neq \lim_\alpha P(x_\alpha) = \phi(P). \qquad \blacksquare$$

In answer to the question of how to identify the evaluations, we can thus say that for $\phi \in \mathrm{cl}B \subset \mathcal{M}_{\mathcal{H}^\infty(B)}$ with ϕ in the fiber over $z^{**} \in B^{**}$, we have $\phi = \delta_{z^{**}}$ if and only if for every net x_α such that $||x_\alpha|| \leq \lambda < 1$ and $\delta_{x_\alpha} \to \phi$ we have that x_α converges polynomial-star to z^{**}.

We now relate these ideas to polynomial reflexivity.

THEOREM 4.5: *The following are equivalent for any Banach space X, and thus if X is reflexive with the approximation property, they are equivalent to polynomial reflexivity:*

(i) *Every bounded net in X which is weak-star convergent in X^{**} is polynomial-star convergent.*
(ii) *Suppose there exist $r < 1$ and $\phi \in \operatorname{cl} rB^{**} \subset \mathcal{M}$ lying in the fiber over $z^{**} \in rB^{**}$. Then $\phi = \delta_{z^{**}}$.*
(iii) *Whenever (x_α) is a net of evaluations converging (in $\mathcal{M}_{\mathcal{H}^\infty(B)}$) to a nonzero element in the fiber over zero, one must have $\|x_\alpha\| \to 1$.*
(iv) *Every N-homogeneous analytic polynomial on X is weakly continuous.*

Proof: The equivalence of (i) and (ii) is a straightforward application of Proposition 4.4, and that (ii) implies (iii) is easy. We show that (iii) implies (iv) and that (iv) implies (i).

To see that (iii) implies (iv), suppose that (iv) fails. Then there is an N and a $P \in \mathcal{P}_N(X)$ which is not weakly continuous at the origin. Take a net which converges weakly to zero but such that $P(x_\alpha)$ does not converge to zero; by scaling down by $\frac{1}{2}$ we may assume that the net is bounded away from the sphere, and since the spectrum is compact there is a cluster point for this scaled down net (thought of as point masses) in $\mathcal{M}_{\mathcal{H}^\infty(B)}$. This cluster point must have a nonzero value at P; thus (iii) fails.

Finally suppose that $\{x_\lambda\}$ converges weak-star to $x^{**} \in X^{**}$. First we consider the subspace $X \oplus [x^{**}]$ of X^{**}; we show that the property (iv) passes from X to this space. If $\{x_\alpha + \beta_\alpha x^{**}\}$ is weakly null then either $\{x_\alpha\}$ is weakly null and $\{\beta_\alpha\}$ tends to zero, or (by passing to a subnet) $\beta_\alpha \to \beta$ and $(x_\alpha) \to -\beta x^{**}$ weakly. We want the canonical extension of any N-homogeneous polynomial on X to tend to zero when evaluated against this net. In the first of the above cases, this is trivial; in the second case, we can expand the associated N-linear form and verify the result by induction.

Now that we know $X \oplus [x^{**}]$ has the property (iv) (whenever X does), we know that the canonical extension of any polynomial P, when restricted to $X \oplus [x^{**}]$, is weakly continuous. Thus $P(x_\lambda) \to \widehat{P}(x^{**})$, and so x_λ converges polynomial-star to x^{**}. ∎

ACKNOWLEDGEMENT: I would like to thank W. B. Johnson for his encouragement and helpful suggestions.

References

[AAD] R. Alencar, R. Aron and S. Dineen, *A reflexive space of holomorphic functions in infinitely many variables*, Proc. Amer. Math. Soc. **90** (1984), 407–411.

[AAF] R. Alencar, R. Aron and G. Fricke, *Tensor products of Tsirelson's space*, Ill. J. Math. **31** (1987), 17–23.

[AB] R. Aron and P. Berner, *A Hahn–Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24.

[ACG] R. Aron, B. Cole and T. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math. **415** (1991), 51–93.

[AHV] R. Aron, C. Herves and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. **52** (1983), 189–204.

[B] B. Beauzamy, *Banach–Saks properties and spreading models*, Math. Scand. **44** (1979), 357–384.

[BS1] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory **7** (1974), 294–299.

[BS2] A. Brunel and L. Sucheston, *On J-convexity and some ergodic super-properties of Banach spaces*, Trans. Amer. Math. Soc. **204** (1975), 79–90.

[CCG] T. Carne, B. Cole and T. Gamelin, *A uniform algebra of analytic functions on a Banach space*, Proc. Amer. Math. Soc. **314** (1989), 639–659.

[CG] B. Cole and T. Gamelin, *Tight uniform algebras and analytic functions*, J. Funct. Anal. **46** (1982), 158–220.

[CGJ] B. Cole, T. Gamelin, and W. B. Johnson, *Analytic disks in fibers over the unit ball of a Banach space*, Mich. Math. J. **39** (1992), 551–569.

[CS] P. Casazza and T. Shura, *Tsirelson's space*, Lecture Notes in Mathematics, Vol. 1363, Springer-Verlag, Berlin, 1989.

[DG] A. M. Davie and T. W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. **106** (1989), 351–358.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, Rhode Island, 1977.

[FJ] J. Farmer and W. B. Johnson, *Polynomial Schur and polynomial Dunford–Pettis properties*, in *Banach Spaces*, Contemporary Mathematics, Vol. 144 (Bor-Luh Lin, ed.), American Mathematical Society, Providence, Rhode Island, 1993.

[JP] J. A. Jaramillo and A. Prieto, *Weak-polynomial convergence on a Banach space*, Proc. Amer. Math. Soc., to appear.

[M] J. Mujica, *Complex Analysis in Banach Spaces,* Notas de Mathematica, Vol. 120, North-Holland, Amsterdam, 1986.

[R1] R. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy,* Ph.D. Thesis, Trinity College, Dublin, 1980.

[R2] R. Ryan, *Weakly compact holomorphic mappings on Banach spaces,* Pac. J. Math. **131** (1988), 179–190.